ON A CONJECTURE OF KONTSEVICH AND SOIBELMAN

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ABSTRACT. We consider a conjecture of Kontsevich and Soibelman which is regarded as a foundation of their theory of motivic Donaldson-Thomas invariants for non-commutative $3d$ Calabi-Yau varieties. We will show that, in some certain cases, the answer to this conjecture is positive.

1. INTRODUCTION

In [10], Kontsevich and Soibelman introduce and give discussions on the motivic Donaldson-Thomas invariants which are defined for non-commutative $3d$ Calabi-Yau varieties and take values in certain Grothendieck groups of algebraic varieties. One of the main objectives of [10] is to define the motivic Hall algebra which generates Toën's notion of the derived Hall algebra (cf. [12]). For $\mathcal{C}$ an ind-constructible triangulated A_∞-category over a field k, the motivic Hall algebra $H(\mathcal{C})$ is constructed to become a graded associative algebra, which admits for each strict sector V an element A_V^{Hall} invertible in the completed motivic Hall algebra and satisfying the Factorization Property. It is believed that, in the case of $3d$ Calabi-Yau category, there is a homomorphism Φ of the motivic Hall algebra into the motivic quantum torus defined in terms of the motivic Milnor fiber of the potential. Then the motivic Donaldson-Thomas invariants appear as the collection of the images of A_V^{Hall} under the homomorphism Φ.

In fact, a central role in the existence of Φ is played by the following conjecture. Assume that k is of characteristic zero. Let F be a formal series on the affine space $\mathbb{A}_k^d = \mathbb{A}_k^{d_1} \times_k \mathbb{A}_k^{d_2} \times_k \mathbb{A}_k^{d_3}$, depending on a constructible way on finitely many extra parameters, such that $F(0,0,0) = 0$ and F has degree zero with respect to the diagonal action of the multiplicative group $\mathbb{G}_{m,k}$ with the weights $(1,-1,0)$. We denote by $X_0(F)$ the set of the zeros of F on $\mathbb{A}_k^d$. Consider be the natural inclusions $i_1 : \mathbb{A}_k^{d_1} \times_k \mathbb{G}_{m,k} \to Z \times_k \mathbb{G}_{m,k}$ and $i_0 : \{0\} \times_k \mathbb{G}_{m,k} \to Z \times_k \mathbb{G}_{m,k}$ for a subset Z containing $X_0(F)$. Consider the motivic Milnor fiber $\mathcal{S}_F$ of F in $\mathcal{M}_{X_0(F) \times_k \mathbb{G}_{m,k}}^{\mathbb{G}_{m,k}}$. Denote by h the function on $\mathbb{A}_k^{d_3}$ defined by $h(z) = F(0,0,z)$. We write $\mathcal{S}_{h,0}$ for the pullback $i_0^* \mathcal{S}_h$. We denote by integral $\int_{\mathbb{A}_k^{d_1}}$ the pushforward of the canonical morphism $\pi : \mathbb{A}_k^{d_1} \times_k \mathbb{G}_{m,k} \to Spec(k) \times_k \mathbb{G}_{m,k}$.

Conjecture 1.1 ([10])**.** *With the previous notations and hypotheses, the following formula holds in $\mathcal{M}_{\mathbb{G}_{m,k}}^{\mathbb{G}_{m,k}}$:*

$$\int_{\mathbb{A}_k^{d_1}} i_1^* \mathcal{S}_F = \mathbb{L}^{d_1} \mathcal{S}_{h,0}.$$

In this paper, we consider the conjecture in some special cases, namely, when F is a compostion of a polynomial in two variables and a pair of two regular functions (Theorem 5.1), or F has the form $F(x,y,z) = g(x,y,z) + h(z)^N$ with N sufficiently

1

large (Theorem 5.6). For these cases, the idea of proof comes from considering formulas of Guibert, Loeser and Merle ([6], [8]) for the motivic Milnor fiber of composite functions or functions of Steenbrink type. The explicit computation of motivic Milnor fiber of a regular function via its Newton polyhedron (suggested in [5]) is the key leading to the positive answers to the cases considered (under certain conditions).

This work was suggested by François Loeser, my advisor, who advised me to consider the conjecture firstly in the case of composition $f(g_1, g_2)$ and encouraged me in each step of proof. I am deeply grateful to him for these, for his suggestions of method approaching to the solution and for his help in preparing the manuscript.

2. Motivic zeta function and Motivic Milnor fiber

Let us recall some basic notations in the theory of motivic integration which will be used in this paper. For references, we follow [1], [2], [4], [5], [6] and [7].

2.1. Arc spaces.
Let X be a smooth algebraic variety of pure dimension n over k. We denote by $\mathcal{L}_m(X)$ the space of arcs of order m also known as the mth jet space on X. It is a k-scheme whose set of K-points, for K a field containing k, is the set of morphisms $\varphi : Spec K[t]/t^{m+1} \to X$. There are canonical morphisms $\mathcal{L}_{m+1}(X) \to \mathcal{L}_m(X)$ which are $\mathbb{A}_k^n$-bundles. The arc space $\mathcal{L}(X)$ is defined as a projective limit of this system. We denote by $\pi_m : \mathcal{L}(X) \to \mathcal{L}_m(X)$ the canonical morphism. There is a canonical $\mathbb{G}_{m,k}$-action on $\mathcal{L}_m(X)$ and on $\mathcal{L}(X)$ given by $a \cdot \varphi(t) = \varphi(at)$.

For an element φ in $K[[t]]$ or in $K[t]/t^{m+1}$, we denote by $ord_t(\varphi)$ the valuation of φ and by $ac(\varphi)$ its first nonzero coefficient with the convention $ac(0) = 0$.

2.2. Motivic zeta function and Motivic Milnor fiber.
Let $g : X \to \mathbb{A}_k^1$ be a function on X and $X_0(g)$ the zero locus of g. For $m \geq 1$, we define

$$\mathfrak{X}_m(g) := \{\varphi \in \mathcal{L}_m(X) \mid ord_t g(\varphi) = m\}.$$

Note that this variety is invariant by the $\mathbb{G}_{m,k}$-action on $\mathcal{L}_m(X)$. Furthermore, g induces a morphism $g_m : \mathfrak{X}_m(g) \to \mathbb{G}_{m,k}$, assigning to a point φ in $\mathcal{L}_m(X)$ the coefficient $ac(g(\varphi))$ of t^m in $g(\varphi(t))$, which we also denote by $ac(g)(\varphi)$. This morphism is a diagonally monomial of weight m with respect to the $\mathbb{G}_{m,k}$-action on $\mathfrak{X}_m(g)$ since $g(s \cdot \varphi) = s^m g_m(\varphi)$. We thus consider the class $[\mathfrak{X}_m(g)]$ of $\mathfrak{X}_m(g)$ in $\mathcal{M}_{X_0(g) \times_k \mathbb{G}_{m,k}}^{\mathbb{G}_{m,k}}$. We now can consider the motivic zeta function

$$Z_g(T) := \sum_{m \geq 1} [\mathfrak{X}_m(g)] \mathbb{L}^{-mn} T^m$$

in $\mathcal{M}_{X_0(g) \times_k \mathbb{G}_{m,k}}^{\mathbb{G}_{m,k}}[[T]]$. Note that $Z_g = 0$ if $g = 0$ on X.

By using a log-resolution of $X_0(g)$, Denef and Loeser proved in [1] and [4] that $Z_g(T)$ is a rational series in $\mathcal{M}_{X_0(g) \times_k \mathbb{G}_{m,k}}^{\mathbb{G}_{m,k}}[[T]]_{sr}$ and they also showed that one can consider the limit $\lim_{T \to \infty} Z_g(T)$ in $\mathcal{M}_{X_0(g) \times_k \mathbb{G}_{m,k}}^{\mathbb{G}_{m,k}}$. Then the motivic Milnor fiber of g is defined as

$$\mathcal{S}_g := -\lim_{T \to \infty} Z_g(T).$$

2.3. **Rational series and their limits.** Let A be one of the rings $\mathbb{Z}[\mathbb{L}, \mathbb{L}^{-1}]$, $\mathbb{Z}[\mathbb{L}, \mathbb{L}^{-1}, (1/(1 - \mathbb{L}^{-i}))_{i>0}]$, $\mathcal{M}_{S \times_k \mathbb{G}_{m,k}^r}^{\mathbb{G}_{m,k}^r}$. We denote by $A[[T]]_{sr}$ the A-submodule of $A[[T]]$ generated by 1 and by finite products of terms $p_{e,i}(T) = \mathbb{L}^e T^i / (1 - \mathbb{L}^e T^i)$ with e in $\mathbb{Z}$ and i in $\mathbb{N}_{>0}$. There is a unique A-linear morphism

$$\lim_{T \to \infty} : A[[T]]_{sr} \to A$$

such that

$$\lim_{T \to \infty} \Big(\prod_{i \in I} p_{e_i, j_i}(T) \Big) = (-1)^{|I|}$$

for every family $((e_i, j_i))_{i \in I}$ in $\mathbb{Z} \times \mathbb{N}_{>0}$ with I finite (possibly empty).

From now on, we will use the following notations

$$\mathbb{R}_{\geq 0}^I := \{a = (a_1, \ldots, a_n) \in \mathbb{R}_{\geq 0}^n \mid a_i = 0 \text{ for } i \notin I\},$$

and

$$\mathbb{R}_{>0}^I := \{a = (a_1, \ldots, a_n) \in \mathbb{R}_{\geq 0}^n \mid a_i = 0 \text{ iff } i \notin I\},$$

for I a subset of $\{1, \ldots, n\}$. The sets $\mathbb{Z}_{\geq 0}^I$, $\mathbb{Z}_{>0}^I$ and $\mathbb{N}_{>0}^I$ are defined similarly.

Let Δ be a rational polyhedral convex cone in $\mathbb{R}_{>0}^I$ and let $\overline{\Delta}$ denote its closure in $\mathbb{R}_{\geq 0}^I$ with I a finite set. Let l and l' be two integer linear forms on $\mathbb{Z}^I$ positive on $\overline{\Delta} \setminus \{0\}$. Let us consider the series

$$S_{\Delta, l, l'}(T) := \sum_{k \in \Delta \cap \mathbb{N}_{>0}^I} \mathbb{L}^{-l'(k)} T^{l(k)}$$

in $\mathbb{Z}[\mathbb{L}, \mathbb{L}^{-1}][[T]]$. In this paper, we will use the following lemmas.

Lemma 2.1 ([5]). *Assume that Δ is open in its linear span and $\overline{\Delta}$ is generated by $(e_1, \ldots, e_m)$ which are part of a $\mathbb{Z}$-basis of the $\mathbb{Z}$-module $\mathbb{Z}^I$. Then the series $S_{\Delta, l, l'}(T)$ lies in $\mathbb{Z}[\mathbb{L}, \mathbb{L}^{-1}][[T]]_{sr}$ and*

$$\lim_{T \to \infty} S_{\Delta, l, l'}(T) = (-1)^{\dim(\Delta)}.$$

Lemma 2.2. *Fix a subset I of $\{1, \ldots, n\}$ with $|I| \geq 2$ and a proper subset K of I. Assume that K is a disjoint union of nonempty subsets $K_1, \ldots, K_m$. Let Δ be a rational polyhedral convex cone in $\mathbb{R}_{>0}^I$ defined by*

$$\sum_{i \in K_j} a_i x_i \leq \sum_{i \in I \setminus K} a_i x_i, \quad j = 1, \ldots, m,$$

with a_i in $\mathbb{N}$, $a_i > 0$ for i in K. If l and l' are integral linear forms positive on $\overline{\Delta} \setminus \{0\}$, then $\lim_{T \to \infty} S_{\Delta, l, l'}(T) = 0$.

Proof. For a subset J of $\{1, \ldots, m\}$, we denote by Δ_J the rational polyhedral convex cone in $\mathbb{R}_{>0}^I$ defined by

$$\sum_{i \in K_j} a_i x_i < \sum_{i \in I \setminus K} a_i x_i \ (j \in J), \quad \sum_{i \in K_j} a_i x_i = \sum_{i \in I \setminus K} a_i x_i \ (j \notin J).$$

Then, Δ is a disjoint union of the cones Δ_J with all J contained in $\{1, \ldots, m\}$ and $\dim(\Delta_J) = |I| + |J| - m$. By Lemma 2.1, $\lim_{T \to \infty} S_{\Delta_J, l, l'}(T) = (-1)^{|I|+|J|-m}$. It deduces that

$$\lim_{T \to \infty} S_{\Delta, l, l'}(T) = \sum_{J \subset \{1, \ldots, m\}} \lim_{T \to \infty} S_{\Delta_J, l, l'}(T) = \sum_{J \subset \{1, \ldots, m\}} (-1)^{|I|+|J|-m} = 0.$$

The lemma is proved.   □

3. Newton polyhedron of a regular function

3.1. **Newton polyhedron.** Let $g(x) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha$ be a polynomial in n variables $x = (x_1, \ldots, x_n)$ such that $g(0) = 0$. We denote by $supp(g)$ the set of exponents α in $\mathbb{N}^n$ with $a_\alpha \neq 0$. The Newton polyhedron Γ of g is the convex hull of $supp(g) + \mathbb{R}^n_{\geq 0}$. For a compact face γ of Γ, we denote by g_γ the following quasihomogenous polynomial

$$g_\gamma(x) = \sum_{\alpha \in \gamma} a_\alpha x^\alpha.$$

We say g is *nondegenerate* with respect to its Newton polyhedron Γ if, for every compact face γ of Γ, the *face* function g_γ is smooth on $\mathbb{G}^n_{m,k}$.

To the Newton polyhedron Γ we associate a function l_Γ which assigns to a vector a in $\mathbb{R}^n_{\geq 0}$ the value $\inf_{b \in \Gamma} \langle a, b \rangle$, with $\langle a, b \rangle$ being the standard inner product of a and b. For a in $\mathbb{R}^n_{\geq 0}$, we denote by γ_a the face of Γ on which the restriction of the function $\langle a, . \rangle$ on $\overline{\Gamma}$ attains its minimum, i.e., $b \in \Gamma$ is in γ_a if and only if

$$\langle a, b \rangle = l_\Gamma(a) = \min_{b \in \Gamma} \langle a, b \rangle.$$

Note that the infimum in the definition of $l_\Gamma(a)$ exists and equals the minimum $\min_{b \in \Gamma} \langle a, b \rangle$. Indeed, by defintion, we can take the infimum on the convex hull of $supp(g)$ instead of Γ, this is a compact subset of $\mathbb{R}^n$. Moreover, the convex hull of the finite set $supp(g)$ is convex and the minimum of a function on a compact convex subset attains at one of its vertices.

For $a = 0$ in $\mathbb{R}^n_{\geq 0}$, $\gamma_a = \Gamma$. If $a \neq 0$, γ_a is a proper face of Γ. Furthermore, γ_a is a compact face of Γ if and only if a is in $\mathbb{R}^n_{>0}$. For any face γ of the Newton polyhedron Γ, we denote by $\sigma(\gamma)$ the cone $\{a \in \mathbb{R}^n_{\geq 0} \mid \gamma_a = \gamma\}$.

Lemma 3.1 ([9]). *Let γ be a proper face of Γ. The following statements hold:*

   (i) *$\sigma(\gamma)$ is a relatively open subset of $\mathbb{R}^n_{\geq 0}$,*
   (ii) *$\overline{\sigma}(\gamma) = \{a \in \mathbb{R}^n_{\geq 0} \mid \gamma_a \supset \gamma\}$ and it is a polyhedral cone,*
   (iii) *The function l_Γ is linear on the cone $\overline{\sigma}(\gamma)$.*

Proof. The Newton polyhedron Γ is in fact the convex hull of a finite set of points, namely $P_1, \ldots, P_s$, and the directions of recession $e_1, \ldots, e_n$, the standard basis of $\mathbb{R}^n$. By [11], one can consider γ as the convex hull of some points of $P_1, \ldots, P_s$ and some directions of recession among $e_1, \ldots, e_n$. For simplicity, we can assume that γ is associated to $P_1, \ldots, P_r$ and $e_1, \ldots, e_k$, with $r < s$ and $k < n$. The cone $\sigma(\gamma)$ is then defined by the following equations and inequations

$$
\begin{aligned}
&\langle a, P_1 \rangle = \langle a, P_2 \rangle = \cdots = \langle a, P_r \rangle, \\
(1) \quad &\langle a, P_1 \rangle < \langle a, P_t \rangle \quad \text{for} \quad r+1 \leq t \leq s, \\
&a_i = 0 \quad \text{for} \quad 1 \leq i \leq k, \\
&a_j > 0 \quad \text{for} \quad k+1 \leq i \leq n,
\end{aligned}
$$

which is a relatively open set. The closure $\overline{\sigma}(\gamma)$ is described similarly as above, replacing the symbol $<$ in the equalities (1) by the symbol $\leq$, and (ii) follows. Furthermore, (ii) implies that there exists a vector $b(\gamma)$ in γ such that $l_\Gamma(a) = \langle a, b(\gamma) \rangle$ for every a in $\overline{\sigma}(\gamma)$. This proves (iii).   □

A *fan* $\mathcal{F}$ is a finite set of rational polyhedral cones such that every face of a cone of $\mathcal{F}$ is also a cone of $\mathcal{F}$, and the intersection of two arbitrary cones of $\mathcal{F}$ is the common face of them. The following lemma shows that, when γ runs over the faces of Γ, $\overline{\sigma}(\gamma)$ form a fan in $\mathbb{R}_{\geq 0}^n$ partitioning $\mathbb{R}_{\geq 0}^n$ into rational polyhedral cones.

Lemma 3.2 ([9])**.** *Let Γ be the Newton polyhedron of the function g. Then the closures $\overline{\sigma}(\gamma)$ of the cones associated to the faces of Γ form a fan in $\mathbb{R}_{\geq 0}^n$. Moreover, the following hold.*

(i) *Fix a proper face γ of Γ. Then the following correspondance is a bijection*

$$\{\text{faces of } \Gamma \text{ that contain } \gamma\} \to \{\text{faces of } \overline{\sigma}(\gamma)\} \ : \ \epsilon \mapsto \overline{\sigma}(\epsilon).$$

(ii) *Let γ_1, γ_2 be faces of Γ. If γ_1 is a facet of γ_2 then $\overline{\sigma}(\gamma_2)$ is a facet of $\overline{\sigma}(\gamma)_1$.*

Recall that a vector a in $\mathbb{R}^n$ is *primitive* if its components are integers whose greatest common divisor is 1. It is easy to see that every face γ of Γ is the intersection of a finite number of facets of Γ. One can also prove that, for any facet of Γ, there exists a unique primitive normal vector in $\mathbb{N}^n \setminus \{0\}$.

Lemma 3.3 ([9])**.** *Assume that the compact face γ is the intersection of the facets $\gamma_1, \ldots, \gamma_r$ with primitive normal vectors $w_1, \ldots, w_r$, respectively. Then we have*

$$\sigma(\gamma) = \Big\{ \sum_{i=1}^r \lambda_i w_i \mid \lambda_i \in \mathbb{R}_{>0} \Big\} \quad and \quad \overline{\sigma}(\gamma) = \Big\{ \sum_{i=1}^r \lambda_i w_i \mid \lambda_i \in \mathbb{R}_{\geq 0} \Big\}.$$

Moreover, $\dim \sigma(\gamma) = \dim \overline{\sigma}(\gamma) = n - \dim \gamma$.

3.2. The canonical partition of $\mathbb{R}_{\geq 0}^{n_1} \times \mathbb{R}_{>0}^{n_2}$ with respect to g. Write $n = n_1 + n_2$ with $n_1 \geq 0$, $n_2 \geq 0$. Let g be a function on $\mathbb{A}_k^n$ which is nondegenerate with respect to the Newton polyhedron Γ of g. Let γ be a compact face of Γ. A proper face ϵ of Γ is said to be *leant* on γ if there exists a subset I of $\{1, \ldots, n\}$ such that

$$\epsilon = \gamma + \mathbb{R}_{\geq 0}^I = \{a + b \mid a \in \gamma, b \in \mathbb{R}_{\geq 0}^I\}.$$

Note that $\dim(\epsilon) = \dim(\gamma) + |I|$. Clearly, the face ϵ is non-compact when I is nonempty. If γ is compact and $\dim(\gamma) = n - 1$, there is no non-compact face of Γ leant on γ because of the reason of dimension.

Lemma 3.4. *Let $\epsilon = \gamma + \mathbb{R}_{\geq 0}^I$ be a face leant on a compact face γ of Γ. If J is a subset of I, then $\epsilon' = \gamma + \mathbb{R}_{\geq 0}^J$ is also a face of Γ leant on γ.*

Proof. If J is a subset of I then $\epsilon' = \gamma + \mathbb{R}_{\geq 0}^J$ is a face of ϵ, hence a face of Γ, because all the faces of Γ form a fan in Γ. $\square$

Notice that if $I = \emptyset$ the face $\gamma + \mathbb{R}_{\geq 0}^I$ reduces to the compact face γ. Suppose that $\epsilon = \gamma + \mathbb{R}_{\geq 0}^I$ is exactly the intersection of the facets $\epsilon_1, \ldots, \epsilon_r$, with $w_i \in \mathbb{N}^n$ being the unique primitive normal vector of ϵ_i, for $i = 1, \ldots, r$. We denote

$$\sigma_{\gamma, I} := \sigma(\epsilon) = cone(w_1, \ldots, w_r).$$

By Lemma 3.3, $\dim(\sigma_{\gamma, I}) = n - |I| - \dim(\gamma)$.

Lemma 3.5. *If $\sigma_{\gamma, I}$ is contained in $\mathbb{R}_{\geq 0}^{n_1} \times \mathbb{R}_{>0}^{n_2}$ and J is a subset of I, then $\sigma_{\gamma, J}$ is contained in $\mathbb{R}_{\geq 0}^{n_1} \times \mathbb{R}_{>0}^{n_2}$. Moreover, $\sigma_{\gamma, I}$ is a face of $\sigma_{\gamma, J}$.*

Proof. Let $\epsilon' = \gamma + \mathbb{R}_{\geq 0}^J$ correspond to the unique primitive normal vectors $v_1, \ldots, v_s$. By Lemma 3.4, since J is a subset of I, ϵ' is a face of ϵ, and hence

$$\{w_1, \ldots, w_r\} \subset \{v_1, \ldots, v_s\}.$$

By assumption, a point of $\sigma_{\gamma, I} \subset \mathbb{R}_{\geq 0}^{n_1} \times \mathbb{R}_{>0}^{n_2}$ is of the form $a = (a_1, \ldots, a_n)$ with $a_i > 0$ for $i = n_1 + 1, \ldots, n$. A point of $\sigma_{\gamma, J}$ is of the form $a + b$ with a in $\sigma_{\gamma, I}$ and $b_i \geq 0$ for every $i = 1, \ldots, n$, hence the first part of the lemma follows. The final statement is a corollary of Lemma 3.2. $\qquad\square$

Lemma 3.6. *Assume that γ is a compact face and $\epsilon = \gamma + \mathbb{R}^I$ is a face of Γ. Then $\sigma_{\gamma, I}$ is contained in $\mathbb{R}_{\geq 0}^{n_1} \times \mathbb{R}_{>0}^{n_2}$ if and only if I is a subset of $\{1, \ldots, n_1\}$.*

Proof. By definition, $\sigma_{\gamma, I} = \sigma(\epsilon) = \{a \in \mathbb{R}_{\geq 0}^n \mid \epsilon_a = \epsilon\}$, where $\epsilon_a = \{b \in \Gamma \mid \langle a, b \rangle = l_\Gamma(a)\}$ for a in $\mathbb{R}_{\geq 0}^n$. Notice that, for every a in $\sigma_{\gamma, I}$, we have the fact that

$$I = \{1 \leq i \leq n \mid a_i = 0\}.$$

This proves the lemma. $\qquad\square$

Fix a compact face γ of Γ. Let M be a maximal element (in the inclusion relation) of the family of the subsets of $\{1, \ldots, n_1\}$ such that $\gamma + \mathbb{R}_{\geq 0}^M$ is a face of Γ (thus, by Lemma 3.6, $\sigma_{\gamma, M}$ is contained in $\mathbb{R}_{\geq 0}^{n_1} \times \mathbb{R}_{>0}^{n_2}$). Then, by Lemma 3.4, for every subset I of M, $\gamma + \mathbb{R}_{\geq 0}^I$ is a face of Γ, and by Lemma 3.5, every cone $\sigma_{\gamma, I}$ is contained in $\mathbb{R}_{\geq 0}^{n_1} \times \mathbb{R}_{>0}^{n_2}$. We thus have proved the following result.

Proposition 3.7. *There exists a canonical fan in $\mathbb{R}_{\geq 0}^{n_1} \times \mathbb{R}_{>0}^{n_2}$ with respect to g partitioning it into the cones $\sigma_{\gamma, I}$, where I runs over the subsets of M, M runs over the maximal subsets of $\{1, \ldots, n_1\}$ such that $\gamma + \mathbb{R}_{\geq 0}^M$ is a face of Γ, and γ runs over the compact faces of Γ.*

Example 3.8. Consider a function g with Γ having a vertex $P = \{(\alpha_1, \ldots, \alpha_n)\}$ as a unique compact face. Then the k-dimensional faces of Γ leant on P have the form

$$P + \mathbb{R}_{\geq 0}^I$$

with I subsets of $\{1, \ldots, n\}$ and $|I| = k$, for $k = 0, \ldots, n - 1$. We deduce from Lemma 3.6 that the canonical partition of $\mathbb{R}_{\geq 0}^{n_1} \times \mathbb{R}_{>0}^{n_2}$ with respect to g is given by the cones $\sigma_{P, I}$, with I subsets of $\{1, \ldots, n_1\}$.

Example 3.9. Consider a function g on $\mathbb{A}_k^3$ defined by

$$g(x, y, z) = x^2 z^2 + xyz^2 + y^3 z^3.$$

Γ has exactly 5 compact faces, namely $P_1 = (2, 0, 2)$, $P_2 = (1, 1, 2)$, $P_3 = (0, 3, 3)$ (0-dimensional faces) and $P_1 P_2$, $P_2 P_3$ (1-dimensional faces).

The faces of Γ leant on P_1 are P_1, $P_1 + \mathbb{R}_{\geq 0}^{\{1\}}$, $P_1 + \mathbb{R}_{\geq 0}^{\{3\}}$ and $P_1 + \mathbb{R}_{\geq 0}^{\{1,3\}}$. The faces leant on P_2 are P_2, $P_2 + \mathbb{R}_{\geq 0}^{\{2\}}$ and $P_2 + \mathbb{R}_{\geq 0}^{\{3\}}$. The faces leant on P_3 are P_3, $P_3 + \mathbb{R}_{\geq 0}^{\{2\}}$, $P_2 + \mathbb{R}_{\geq 0}^{\{3\}}$ and $P_2 + \mathbb{R}_{\geq 0}^{\{2,3\}}$. The faces leant on $P_1 P_2$ are $P_1 P_2$, $P_1 P_2 + \mathbb{R}_{\geq 0}^{\{1\}}$ and $P_1 P_2 + \mathbb{R}_{\geq 0}^{\{3\}}$. The faces leant on $P_2 P_3$ are $P_2 P_3$, $P_2 P_3 + \mathbb{R}_{\geq 0}^{\{2\}}$ and $P_2 P_3 + \mathbb{R}_{\geq 0}^{\{3\}}$.

By Lemma 3.6, the cones contained in $\mathbb{R}_{\geq 0}^2 \times \mathbb{R}_{>0}^1$ are $\sigma_{P_1, \emptyset}$, $\sigma_{P_1, \{1\}}$, $\sigma_{P_2, \emptyset}$, $\sigma_{P_2, \{2\}}$, $\sigma_{P_3, \emptyset}$, $\sigma_{P_3, \{2\}}$, $\sigma_{P_1 P_2, \emptyset}$, $\sigma_{P_1 P_2, \{1\}}$, $\sigma_{P_2 P_3, \emptyset}$ and $\sigma_{P_2 P_3, \{2\}}$. The closures of these cones form a canonical fan of $\mathbb{R}_{\geq 0}^2 \times \mathbb{R}_{>0}^1$.

Remark 3.10. In the case $n_1 = 0$, we reduce to the work by Guibert (cf. [5]). More clearly, for each compact face γ of Γ, all the maximal subsets M of $\{1, \ldots, n\}$, of which $\gamma + \mathbb{R}_{\geq 0}^M$ is a face of Γ and $\sigma_{\gamma,M} \subset \mathbb{R}_{>0}^n$, are empty.

4. Computation of $i_1^* \mathcal{S}_g$ and $\int_{\mathbb{A}_k^{d_1}} i_1^* \mathcal{S}_g$

Consider a regular function g on $\mathbb{A}_k^n$. We assume that g is nondegenerate with respect to its Newton polyhedron Γ. Denote by i_1 the natural inclusion $\mathbb{A}_k^{n_1} \hookrightarrow \mathbb{A}_k^n$ or $\mathbb{A}_k^{n_1} \times_k \mathbb{G}_{m,k} \hookrightarrow \mathbb{A}_k^n \times_k \mathbb{G}_{m,k}$.

4.1. The motivic zeta function $Z_g(T)$.
We identify the arc space $\mathcal{L}(\mathbb{A}_k^n)$ with the space of formal power series $k[[t]]^n$ via the system of coordinates $x_1, \ldots, x_n$. For every arc $\varphi \in \mathcal{L}(\mathbb{A}_k^n)$ we note $ord_t x(\varphi) = (ord_t x_1(\varphi), \ldots, ord_t x_n(\varphi))$. For every $m \in \mathbb{N}_{>0}$ and $a \in \mathbb{N}^n$ we set

$$\mathfrak{X}_{a,m}(g) = \mathfrak{X}_m(g) \cap \mathfrak{X}_a,$$

where the spaces $\mathfrak{X}_m(g)$ and $\mathfrak{X}_a$ are defined as follows

$$\mathfrak{X}_m(g) = \{\varphi \in \mathcal{L}_m(\mathbb{A}_k^n) \mid ord_t g(\varphi) = m\},$$
$$\mathfrak{X}_a = \{\varphi \in \mathcal{L}(\mathbb{A}_k^n) \mid ord_t x(\varphi) = a\}.$$

It is clear that $\mathfrak{X}_{a,m}(g)$ is a variety over $X_0(g) \times_k \mathbb{G}_{m,k}$ in which the morphism to $X_0(g)$ is induced by the canonical morphism $\mathcal{L}_m(\mathbb{A}_k^n) \to \mathbb{A}_k^n$ and the morphism to $\mathbb{G}_{m,k}$ is the morphism $ac(g)$. Note that $\mathfrak{X}_{a,m}(g)$ is invariant by the $\mathbb{G}_{m,k}$-action on $\mathcal{L}_m(\mathbb{A}_k^n)$.

For every $a \in \mathbb{N}^n$ and $\varphi \in \mathfrak{X}_a$, $ord_t g(\varphi) \geq l_\Gamma(a)$ by the definition of l_Γ. Furthermore, $\mathfrak{X}_m(g)$ can be expressed as a disjoint union $\bigcup_{a \in \mathbb{N}^n} \mathfrak{X}_{a,m}(g)$ of the subspaces $\mathfrak{X}_{a,m}(g)$ for a in $\mathbb{N}^n$. Then the motivic zeta function $Z_g(T)$ of g can be written in the following form

$$Z_g(T) = \sum_{a \in \mathbb{N}^n} \sum_{m \geq l_\Gamma(a)} [\mathfrak{X}_{a,m}(g)] \mathbb{L}^{-nm} T^m$$

$$= \sum_{a \in \mathbb{N}^n} \left([\mathfrak{X}_{a,l_\Gamma(a)}(g)] \mathbb{L}^{-nl_\Gamma(a)} T^{l_\Gamma(a)} + \sum_{m \geq l_\Gamma(a)+1} [\mathfrak{X}_{a,m}(g)] \mathbb{L}^{-nm} T^m \right)$$

$$=: Z^0(T) + Z^1(T).$$

By Lemma 3.2, there is a canonical partition of $\mathbb{R}_{\geq 0}^n$ into the rational polyhedral cones $\sigma(\gamma)$ with γ running over the proper faces of $\bar{\Gamma}$. We deduce that

$$Z^0(T) = \sum_\gamma \sum_{a \in \sigma(\gamma)} [\mathfrak{X}_{a,l_\Gamma(a)}(g)] \mathbb{L}^{-nl_\Gamma(a)} T^{l_\Gamma(a)},$$

$$Z^1(T) = \sum_\gamma \sum_{a \in \sigma(\gamma)} \sum_{k \geq 1} [\mathfrak{X}_{a,l_\Gamma(a)+k}(g)] \mathbb{L}^{-n(l_\Gamma(a)+k)} T^{l_\Gamma(a)+k},$$

where the sum $\sum_\gamma$ runs over the proper faces γ of Γ.

4.2.
Assume that g satisfies the additional condition that $\mathbb{A}_k^{n_1}$ is naturally included in $X_0(g)$ via the morphism i_1. To compute $i_1^* Z_g(T)$, we consider the canonical fan in $\mathbb{R}_{\geq 0}^{n_1} \times \mathbb{R}_{>0}^{n_2}$ with respect to g. Denote by Γ_c the set of compact faces of Γ, by $\mathfrak{M}_\gamma$ the set of maximal subsets M of $\{1, \ldots, n_1\}$ such that $\gamma + \mathbb{R}_{\geq 0}^M$ is a face of Γ. By Proposition 3.7, we can partition $\mathbb{R}_{\geq 0}^{n_1} \times \mathbb{R}_{>0}^{n_2}$ into the cones $\sigma_{\gamma,I}$, with I subsets

of M, M in $\mathfrak{M}_\gamma$ and γ in Γ_c. Assume that $\mathfrak{M}_\gamma = \{M_1, \ldots, M_p\}$. We denote by $\mathfrak{S}_\gamma$ the family of subsets of one of the sets $M_1, \ldots, M_p$. Then we have

$$i_1^* Z^0(T) = i_1^* \Big(\sum_{\gamma \in \Gamma_c} \sum_{I \in \mathfrak{S}_\gamma} \sum_{a \in \sigma_{\gamma, I}} [\mathfrak{X}_{a, l_\Gamma(a)}(g)] \mathbb{L}^{-n l_\Gamma(a)} T^{l_\Gamma(a)} \Big),$$

$$i_1^* Z^1(T) = i_1^* \Big(\sum_{\gamma \in \Gamma_c} \sum_{I \in \mathfrak{S}_\gamma} \sum_{a \in \sigma_{\gamma, I}} \sum_{k \geq 1} [\mathfrak{X}_{a, l_\Gamma(a)+k}(g)] \mathbb{L}^{-n(l_\Gamma(a)+k)} T^{l_\Gamma(a)+k} \Big).$$

4.3. **Class of $\mathfrak{X}_{a,m}(g)$.** For each compact γ and $I \in \mathfrak{S}_\gamma$, consider the morphism $g_{\gamma, I} : \mathbb{G}_{m,k}^n \to X_0(g) \times_k \mathbb{G}_{m,k}$ given by $g_{\gamma, I}(\xi_1, \ldots, \xi_n) = \big((\hat{\xi}_1, \ldots, \hat{\xi}_n), g_\gamma(\xi_1, \ldots, \xi_n) \big)$, where $\hat{\xi}_i$ is defined as follows

$$\hat{\xi}_i = \begin{cases} \xi_i & \text{if } i \in I \\ 0 & \text{otherwise.} \end{cases}$$

This morphism $g_{\gamma, I}$ being $\mathbb{G}_{m,k}$-equivariant in an obvious manner, defines a class in $\mathcal{M}_{X_0(g) \times_k \mathbb{G}_{m,k}}^{\mathbb{G}_{m,k}}$, which we denote by $\Phi_{\gamma, I}$.

Observe that the morphism $\mathbb{G}_{m,k}^n \to \mathbb{G}_{m,k}$, which is the composition of $g_{\gamma, I}$ with the second projection, is a locally trivial fibration. It deduces that $\Phi_{\gamma, I}$ is equal in $\mathcal{M}_{X_0(g) \times_k \mathbb{G}_{m,k}}^{\mathbb{G}_{m,k}}$ to the class $[g_{\gamma, I}^{-1}(1) \times_k \mathbb{G}_{m,k}]$ of the morphism

$$g_{\gamma, I}^{-1}(1) \times_k \mathbb{G}_{m,k} \to X_0(g) \times_k \mathbb{G}_{m,k}, \ \ \big((\xi_1, \ldots, \xi_n), t \big) \mapsto \big((\hat{\xi}_1, \ldots, \hat{\xi}_n), t \big).$$

We denote by $\Psi_{\gamma, I}$ the class in $\mathcal{M}_{X_0(g) \times_k \mathbb{G}_{m,k}}^{\mathbb{G}_{m,k}}$ of the morphism

$$g_{\gamma, I}^{-1}(0) \times_k \mathbb{G}_{m,k} \times_k \mathbb{G}_{m,k} \to X_0(g) \times_k \mathbb{G}_{m,k},$$

given by $\big((\xi_1, \ldots, \xi_n), s, t \big) \mapsto \big((\hat{\xi}_1, \ldots, \hat{\xi}_n), t \big)$.

Lemma 4.1. *The following formulas hold in $\mathcal{M}_{X_0(g) \times_k \mathbb{G}_{m,k}}^{\mathbb{G}_{m,k}}$ for every a in $\sigma_{\gamma, I}$.*

   (i) $[\mathfrak{X}_{a, l_\Gamma(a)}(g)] = \Phi_{\gamma, I} \mathbb{L}^{n l_\Gamma(a) - s(a)}$.
   (ii) $[\mathfrak{X}_{a, l_\Gamma(a)+k}(g)] = \Psi_{\gamma, I} \mathbb{L}^{n(l_\Gamma(a)+k) - s(a)}$.

Proof. In the case where $a_i \leq l_\Gamma(a)$ for every $i = 1, \ldots, n$, there is a way to prove (i) directly as follows. An element $\varphi(t)$ of $\mathfrak{X}_{a, l_\Gamma(a)}(g)$ has the form $\varphi(t) = (x_1(t), \ldots, x_n(t))$, where $x_i(t) = \sum_{m=a_i}^{l_\Gamma(a)} c_{i,m} t^m$ with $c_{i,a_i} \neq 0$ for $i = 1, \ldots, n$. Note that the coefficient of $t^{l_\Gamma(a)}$ in $g(\varphi(t))$ is equal to

$$\frac{1}{l_\Gamma(a)!} \cdot \frac{d^{l_\Gamma(a)} g(\varphi(t))}{dt^{l_\Gamma(a)}} \Big|_{t=0} = \frac{1}{l_\Gamma(a)!} \cdot \frac{d^{l_\Gamma(a)} g_\gamma(\varphi(t))}{dt^{l_\Gamma(a)}} \Big|_{t=0}$$
$$= g_\gamma(c_{1,a_1}, \ldots, c_{n,a_n})$$

which is nonzero for every a in $\sigma_{\gamma, I}$. One deduces from this that $\mathfrak{X}_{a, l_\Gamma(a)}(g)$ is isomophic to $\mathbb{G}_{m,k}^n \times_k \mathbb{A}_k^{n l_\Gamma(a) - s(a)}$ via the map

$$\varphi(t) \mapsto \big((c_{i,a_i})_{1 \leq i \leq n}, (c_{i,m})_{1 \leq i \leq n, a_i+1 \leq m \leq l_\Gamma(a)} \big).$$

Then the part (i) follows.

Observe that (i) and (ii) also can be deduced from proofs of Guibert in [5] (see [5], Lemma 2.1.1). $\qquad \square$

4.4. **An explicit formula for $i_1^* \mathcal{S}_g$.** Assume that $X_0(g)$ contains $\mathbb{A}_k^{n_1} \times_k \{0\}$. We deduce from Lemma 4.1 that

$$i_1^* Z^0(T) = \sum_{\gamma \in \Gamma_c} \sum_{I \in \mathfrak{S}_\gamma} \sum_{a \in \sigma_{\gamma,I}} i_1^* \Phi_{\gamma,I} \mathbb{L}^{-s(a)} T^{l_\Gamma(a)},$$

and

$$i_1^* Z^1(T) = i_1^* \Big(\sum_{\gamma \in \Gamma_c} \sum_{I \in \mathfrak{S}_\gamma} \sum_{a \in \sigma_{\gamma,I}} \Psi_{\gamma,I} \mathbb{L}^{-s(a)} T^{l_\Gamma(a)} \sum_{k \geq 1} \mathbb{L}^{-k} T^k \Big)$$

$$= \frac{\mathbb{L}^{-1} T}{1 - \mathbb{L}^{-1} T} \sum_{\gamma \in \Gamma_c} \sum_{I \in \mathfrak{S}_\gamma} \sum_{a \in \sigma_{\gamma,I}} i_1^* \Psi_{\gamma,I} \mathbb{L}^{-s(a)} T^{l_\Gamma(a)}.$$

Proposition 4.2. *Assume that g is a regular function on $\mathbb{A}_k^n$ nondegenerate with respect to its Newton polyhedron Γ, that no vertex of Γ lies in a coordinate m-plane $(m = 1, \ldots, n-1)$, and that $X_0(g)$ contains $\mathbb{A}_k^{n_1} \times_k \{0\}$. With the previous notations, the following formula holds in $\mathcal{M}_{\mathbb{A}_k^{n_1} \times_k \mathbb{G}_{m,k}}^{\mathbb{G}_{m,k}}$:*

$$i_1^* \mathcal{S}_g = \sum_{\gamma \in \Gamma_c} (-1)^{n+1-\dim(\gamma)} \sum_{I \in \mathfrak{S}_\gamma} (-1)^{|I|} [\mathbb{A}_k^{n_1} \times_{X_0(g)} (\Phi_{\gamma,I} - \Psi_{\gamma,I})].$$

Proof. By the assumption that no vertex of Γ lies in a coordinate m-plane, $m = 1, \ldots, n-1$, we have

$$\lim_{T \to \infty} \sum_{a \in \sigma_{\gamma,I}} \Phi_{\gamma,I} \mathbb{L}^{-s(a)} T^{l_\Gamma(a)} = (-1)^{n-|I|-\dim(\gamma)} \Phi_{\gamma,I},$$

$$\lim_{T \to \infty} \sum_{a \in \sigma_{\gamma,I}} \Psi_{\gamma,I} \mathbb{L}^{-s(a)} T^{l_\Gamma(a)} = (-1)^{n-|I|-\dim(\gamma)} \Psi_{\gamma,I}$$

since $\dim(\sigma_{\gamma,I}) = n - |I| - \dim(\gamma)$. It follows that

$$\lim_{T \to \infty} i_1^* Z^0(T) = \sum_{\gamma \in \Gamma_c} (-1)^{n-\dim(\gamma)} \sum_{I \in \mathfrak{S}_\gamma} (-1)^{|I|} i_1^* \Phi_{\gamma,I},$$

and

$$\lim_{T \to \infty} i_1^* Z^1(T) = \sum_{\gamma \in \Gamma_c} (-1)^{n+1-\dim(\gamma)} \sum_{I \in \mathfrak{S}_\gamma} (-1)^{|I|} i_1^* \Psi_{\gamma,I}.$$

Then the proposition is proved. $\square$

Example 4.3 (cf. Example 3.8). In the case Γ_g has a unique compact face $P = (\alpha_1, \ldots, \alpha_n)$, the classes $\Psi_{P,I}$ vanish. If we assume that $\alpha_i > 0$ for every $i = 1, \ldots, n$, we have

$$i_1^* \mathcal{S}_g = (-1)^{n+1} \sum_{I \subset \{1, \ldots, n_1\}} (-1)^{|I|} [\mathbb{A}_k^{n_1} \times_{X_0(g)} \Phi_{P,I}].$$

Remark 4.4. In fact, by using Lemma 2.2, we can consider the formula for $i_1^* \mathcal{S}_g$ without assuming that no vertex of Γ lies in a coordinate plane as in Proposition 4.2. Let γ be a compact face of Γ. Assume that γ lies in the coordinate plane defined by $\{x_i = 0, i \in I_\gamma\}$ (I_γ possibly empty). We denote by $\mathfrak{S}_\gamma^>$ the set $\{I \in \mathfrak{S}_\gamma \mid I \supset I_\gamma\}$. Then we have

Proposition 4.5. *Assume that g is a regular function on $\mathbb{A}_k^n$ nondegenerate with respect to its Newton polyhedron Γ and that $X_0(g)$ contains $\mathbb{A}_k^{n_1} \times_k \{0\}$. Then the following formula holds in $\mathcal{M}_{\mathbb{A}_k^{n_1} \times_k \mathbb{G}_{m,k}}^{\mathbb{G}_{m,k}}$:*

$$(2) \qquad i_1^* \mathcal{S}_g = \sum_{\gamma \in \Gamma_c} (-1)^{n+1-\dim(\gamma)} \sum_{I \in \mathfrak{S}_\gamma^>} (-1)^{|I|} [\mathbb{A}_k^{n_1} \times_{X_0(g)} (\Phi_{\gamma,I} - \Psi_{\gamma,I})].$$

Proof. Let γ be a compact face of Γ. For a proper subset I (possibly empty) of I_γ, the cone $\sigma'_{\gamma,I} := \sigma_{\gamma,I} \cap \{a | \mathcal{X}_{a,l_\Gamma(a)} \neq \emptyset\}$ is a rational polyhedral cone in $\mathbb{R}_{>0}^{I^c}$ (by Lemma 3.6) defined by some inequalities of the form as in Lemma 2.2. Here, I^c denotes the complement of I in $\{1, \ldots, n\}$. More precisely, the cone $\sigma'_{\gamma,I}$ is defined by

$$x_j \leq l_\Gamma(x_1, \ldots, x_n), \ j \in I_\gamma \setminus I,$$

where, in the expression of $l_\Gamma(x_1, \ldots, x_n)$ in the canonical basis, the coefficients of x_i with i in I_γ^c are unique nonzero ones. By Lemma 2.2, we have

$$\lim_{T \to \infty} \sum_{a \in \sigma_{\gamma,I}} \Phi_{\gamma,I} \mathbb{L}^{-s(a)} T^{l_\Gamma(a)} = \lim_{T \to \infty} \sum_{a \in \sigma_{\gamma,I}} \Psi_{\gamma,I} \mathbb{L}^{-s(a)} T^{l_\Gamma(a)} = 0.$$

For I in $\mathfrak{S}_\gamma$ containing I_γ, the limits are nonzero by Lemma 2.1, and the arguments on these cases are similar to those of the proof of Proposition 4.2. $\qquad \square$

Corollary 4.6 ([5]). *Let $\Gamma_c^>$ be the set of compact faces of Γ, which are not contained in any coordinate plane. If g is nondegenerate with respect to Γ, then*

$$\mathcal{S}_{g,0} = (-1)^{n-1} \sum_{\gamma \in \Gamma_c^>} (-1)^{\dim(\gamma)} [\{0\} \times_{X_0(g)} (\Phi_{\gamma,I} - \Psi_{\gamma,I})]$$

holds in $\mathcal{M}_{\mathbb{G}_{m,k}}^{\mathbb{G}_{m,k}}$.

Proof. (See Remark 3.10) Consider the formula (2) in the case $n_1 = 0$. Then the natural inclusion $i_1 : \mathbb{A}_k^{n_1} \hookrightarrow \mathbb{A}_k^n$ reduces to the inclusion $i_0 : \{0\} \hookrightarrow \mathbb{A}_k^n$. Moreover, in this case, by Lemma 3.6, for every compact face γ of Γ, we have $\mathfrak{S}_\gamma = \{\emptyset\}$. Hence the equality (2) implies this proposition. Observe that this formula was already obtained by Guibert (cf. [5], Proposition 2.1.6). $\qquad \square$

4.5. Write $g(x) = \sum_{(\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n} a_{\alpha_1 \ldots \alpha_n} x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Assume that g is nondegenerate with respect to Γ and satisfies the following condition

$$(3) \qquad\qquad \alpha_1 + \cdots + \alpha_{n_1} = \alpha_{n_1+1} + \cdots + \alpha_n$$

for every $(\alpha_1, \ldots, \alpha_n)$ in $\mathbb{N}^n$. Because $supp(g)$ lies on the hyperplane $H = \{\alpha_1 + \cdots + \alpha_{n_1} = \alpha_{n_1+1} + \cdots + \alpha_n\}$ in $\mathbb{R}_{\geq 0}^n$, the compact faces of Γ are contained in H. Moreover, for the same reason, for each compact γ, the non-compact faces of Γ leant on γ exist. Note that, in this case, $X_0(g)$ contains $\mathbb{A}_k^{n_1} \times_k \{0\}$.

Lemma 4.7. *Assume that $g(x) = \sum_{(\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n} a_{\alpha_1 \ldots \alpha_n} x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is nondegenerate with respect to Γ and the equalities (3) are satisfied for all $(\alpha_1, \ldots, \alpha_n)$ in $\mathbb{N}^n$ such that $a_{\alpha_1 \ldots \alpha_n} \neq 0$. Then, for every compact face γ of Γ, we have $|\mathfrak{M}_\gamma| = 1$ and the unique element of $\mathfrak{M}_\gamma$ is nonempty.*

Proof. Let γ be a compact face of Γ. Assume that $\gamma + \mathbb{R}_{\geq 0}^I$ is a face of Γ. Then, by Lemma 3.6, the cone $\sigma_{\gamma, I}$ is contained in $\mathbb{R}_{\geq 0}^{n_1} \times \mathbb{R}_{>0}^{n_2}$ if and only if I is contained in $\{1, \ldots, n_1\}$. Furthermore, we claim that if $\gamma + \mathbb{R}_{\geq 0}^I$ and $\gamma + \mathbb{R}_{\geq 0}^J$ are faces leant on γ such that the corresponding cones $\sigma_{\gamma, I}$ and $\sigma_{\gamma, J}$ are both contained in $\mathbb{R}_{\geq 0}^{n_1} \times \mathbb{R}_{>0}^{n_2}$, then so is $\gamma + \mathbb{R}_{\geq 0}^{I \cup J}$. Indeed, the equalities (3) for $(\alpha_1, \ldots, \alpha_n)$ in $\mathbb{N}^n$ guarantee that if I and J are contained in $\{1, \ldots, n_1\}$, the intersection of $\gamma + \mathbb{R}_{\geq 0}^{I \cup J}$ with the interior of Γ is empty. This together with the fact that $\gamma + \mathbb{R}_{\geq 0}^I$ and $\gamma + \mathbb{R}_{\geq 0}^J$ are faces of Γ show that $\gamma + \mathbb{R}_{\geq 0}^{I \cup J}$ is a face of Γ leant on γ such that $\sigma_{\gamma, I \cup J}$ is contained in $\mathbb{R}_{\geq 0}^{n_1} \times \mathbb{R}_{>0}^{n_2}$.

As a consequence of the above claim, for each compact face γ of Γ, there exists a unique maximal subset M of $\{1, \ldots, n\}$ such that $\gamma + \mathbb{R}_{\geq 0}^M$ is a face of Γ, which is leant on γ, and $\sigma_{\gamma, M}$ is contained in $\mathbb{R}_{\geq 0}^{n_1} \times \mathbb{R}_{>0}^{n_2}$. The nonemptyness of the set M follows from the fact that $supp(g)$ lies on the hyperplane H. $\qquad\square$

Corollary 4.8. *Assume that $g(x) = \sum_{(\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_{>0}^n} a_{\alpha_1 \ldots \alpha_n} x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is nondegenerate with respect to Γ and $\alpha_1 + \cdots + \alpha_{n_1} = \alpha_{n_1+1} + \cdots + \alpha_n$ for every $(\alpha_1, \ldots, \alpha_n)$ in $\mathbb{N}_{>0}^n$. Then $\int_{\mathbb{A}_k^{d_1}} i_1^* \mathcal{S}_g$ vanishes in $\mathcal{M}_{\mathbb{G}_{m,k}}^{\mathbb{G}_{m,k}}$.*

Proof. Let γ be a compact face of Γ. By Lemma 4.7, the set $\mathfrak{M}_\gamma$ has a unique element and this element is nonempty. Assume $\mathfrak{M}_\gamma = \{M\}$ with $|M| \geq 1$. Notice that $\int_{\mathbb{A}_k^{d_1}} i_1^* \Phi_{\gamma, I}$ and $\int_{\mathbb{A}_k^{d_1}} i_1^* \Psi_{\gamma, I}$ depend only on γ, not on I contained in M. Because of the fact that, if $m \geq 1$, $\sum_{j=0}^m (-1)^j \binom{m}{j} = 0$, one deduces that

$$\sum_{I \subset M} (-1)^{|I|} \int_{\mathbb{A}_k^{d_1}} i_1^* (\Phi_{\gamma, I} - \Psi_{\gamma, I}) = 0.$$

By Proposition 4.2, the image $\int_{\mathbb{A}_k^{d_1}} i_1^* \mathcal{S}_g$ of $\mathcal{S}_g$ vanishes in $\mathcal{M}_{\mathbb{G}_{m,k}}^{\mathbb{G}_{m,k}}$. $\qquad\square$

5. KONTSEVICH-SOIBELMAN'S CONJECTURE

In this section, we will show that, under certain assumptions, Conjecture 1.1 is true.

5.1. Composition with a polynomial in two variables.
We consider the conjecture of Kontsevich and Soibelman (Conjecture 1.1) in the case where F has the form $F(x, y, z) = f(g_1(x, y), g_2(z))$, where f is a polynomial in two variables with $f(0, y)$ nonzero of positive degree, g_1 is a function on $\mathbb{A}_k^{d_1} \times_k \mathbb{A}_k^{d_2}$ such that $g_1(tx, t^{-1}y) = g_1(x, y)$, $g_1(0, 0) = 0$, and g_2 is a regular function on $\mathbb{A}_k^{d_3}$. Denote $\mathbf{g} = g_1 \times g_2$ and $X_0(\mathbf{g}) = \{(x, y, z) \mid g_1(x, y) = g_2(z) = 0\}$. Recall that, in this case, $h(z) = f(0, g_2(z))$.

Theorem 5.1. *Assume that f is a polynomial in two variables with $f(0, y)$ nonzero of positive degree. Let g_1 be a regular function on $\mathbb{A}_k^{d_1} \times_k \mathbb{A}_k^{d_2}$ nondegenerate with respect to its Newton polyhedron Γ_{g_1} such that $g_1(0, 0) = 0$, no vertex of Γ_{g_1} lies in a coordinate plane, and $g_1(tx, t^{-1}y) = g_1(x, y)$ for every t in $\mathbb{G}_{m,k}$. Let g_2 be a regular function on $\mathbb{A}_k^{d_3}$. Then, the following formula*

$$\int_{\mathbb{A}_k^{d_1}} i_1^* \mathcal{S}_{f \circ \mathbf{g}} = \mathbb{L}^{d_1} \mathcal{S}_{h,0}$$

holds in $\mathcal{M}_{\mathbb{G}_{m,k}}^{\mathbb{G}_{m,k}}$. In other words, in this case, Conjecture 1.1 is true.

Proof. In [8], Guibert, Loeser and Merle consider the motivic Milnor fiber of a composition of the form $f(g_1, g_2)$ where g_1 and g_2 have no variable in common and f is a polynomial in $k[x, y]$ such that $f(0, y)$ is nonzero of positive degree. To describe it, they used the generalized convolution operators Ψ_Q defined in [7] and the tree of contact $\tau(f, 0)$ constructed in terms of Puiseux expansions by Guibert ([5]), here 0 is the origin of $\mathbb{A}_k^d$ with $d = d_1 + d_2 + d_3$. To any rupture vertex v of $\tau(f, 0)$ one attaches a weighted homogeneous polynomial Q_v in $k[X, Y]$. The virtual objects A_v are defined inductively in terms of the tree of contact $\tau(f, 0)$ and A_{v_0}, where v_0 is the first (extended) rupture vertex of the tree and A_{v_0} depends only on g. Let i be the inclusion of $X_0(\mathbf{g}) \times_k \mathbb{G}_{m,k}$ in $Z \times_k \mathbb{G}_{m,k}$. Let m_0 be the order of 0 as a root of $f(0, y)$. By the main theorem of [8], the following formula

$$i^* \mathcal{S}_{f \circ \mathbf{g}} = \mathcal{S}_{g_2^{m_0}} ([X_0(g_1)]) - \sum_v \Psi_{Q_v}(A_v)$$

holds in $\mathcal{M}_{X_0(\mathbf{g}) \times_k \mathbb{G}_{m,k}}^{\mathbb{G}_{m,k}}$, where the sum runs over the augmented set of rupture vertices of the tree $\tau(f, 0)$. Take the operator $\int_{\mathbb{A}_k^{d_1}} i_1^*$ for two sides of the formula, we have

$$\int_{\mathbb{A}_k^{d_1}} i_1^* \mathcal{S}_{f \circ \mathbf{g}} = \int_{\mathbb{A}_k^{d_1}} i_1^* \mathcal{S}_{g_2^{m_0}} ([X_0(g_1)]) - \sum_v \int_{\mathbb{A}_k^{d_1}} i_1^* \Psi_{Q_v}(A_v).$$

We claim that, with previous notations and hypotheses, the formula

$$\int_{\mathbb{A}_k^{d_1}} i_1^* \mathcal{S}_{g_2^{m_0}} ([X_0(g_1)]) = \mathbb{L}^{d_1} \mathcal{S}_{h,0}$$

holds in $\mathcal{M}_{\mathbb{G}_{m,k}}^{\mathbb{G}_{m,k}}$. Indeed, as in [6], proof of Theorem 5.18, one can check that

$$i^* \mathcal{S}_{g_2^{m_0}} ([X_0(g_1)]) = [g_1^{-1}(0)] \boxtimes \mathcal{S}_{g_2^{m_0}}.$$

By the hypotheses on g_1 and the fact that $i_1(\mathbb{A}_k^{d_1}) \cap g_2^{-1}(0) = \{0\}$, we have $i_1^*[g_1^{-1}(0)] = [\mathbb{A}_k^{d_1}] = \mathbb{L}^{d_1}$ and $i_1^* \mathcal{S}_{g_2^{m_0}} = i_0^* \mathcal{S}_{g_2^{m_0}} = \mathcal{S}_{g_2^{m_0}, 0}$. One deduces that

$$i_1^* \mathcal{S}_{g_2^{m_0}} ([X_0(g_1)]) = i_1^* \Big([g_1^{-1}(0)] \boxtimes \mathcal{S}_{g_2^{m_0}} \Big) = \mathbb{L}^{d_1} \mathcal{S}_{g_2^{m_0}, 0}.$$

By definition of h, $\mathcal{S}_{g_2^{m_0}, 0} = \mathcal{S}_{h,0}$, the claim then follows. So, in order to finish the proof of Theorem 5.1, it suffices to prove that $\int_{\mathbb{A}_k^{d_1}} i_1^* \Psi_{Q_v}(A_v) = 0$ for every (extended) rupture vertex v of $\tau(f, 0)$.

Let v_0 be the first (extended) rupture vertex of the tree of contact $\tau(f, p)$. As in [8], the virtual object A_{v_0} in $\mathcal{M}_{X_0(\mathbf{g}) \times_k (\mathbb{A}_k^1 \times_k \mathbb{G}_{m,k})}^{\mathbb{G}_{m,k}}$ is defined by $A_{v_0} := \mathcal{S}'_{g_2} \boxtimes \mathcal{S}_{g_1}$, where $\mathcal{S}'_{g_2}$ is an element in $\mathcal{M}_{X_0(g_2) \times_k \mathbb{A}_k^1}^{\mathbb{G}_{m,k}}$ which is the "disjoint sum" of $\mathcal{S}_{g_2}$ in $\mathcal{M}_{X_0(g_2) \times_k \mathbb{G}_{m,k}}^{\mathbb{G}_{m,k}}$ and $X_0(g_2)$ in $\mathcal{M}_{X_0(g_2)}$.

Lemma 5.2. *Assume that g_1 is a regular function on $\mathbb{A}_k^{d_1} \times_k \mathbb{A}_k^{d_2}$ nondegenerate with respect to its Newton polyhedron Γ_{g_1} such that $g_1(0, 0) = 0$, no vertex of Γ_{g_1} lies in a coordinate plane, and $g_1(tx, t^{-1}y) = g_1(x, y)$ for every t in $\mathbb{G}_{m,k}$. Let g_2 be a regular function on $\mathbb{A}_k^{d_3}$. Then $\int_{\mathbb{A}_k^{d_1}} i_1^* \Psi_Q(A_{v_0})$ vanishes in $\mathcal{M}_{\mathbb{G}_{m,k}}^{\mathbb{G}_{m,k}}$ for every quasi-homogeneous polynomial Q.*

Proof. The assumptions on g_1 mean that we can write g_1 in the form

$$g_1(x,y) = \sum_{(\alpha,\beta) \in \mathbb{N}_{>0}^{d_1+d_2}} a_{\alpha\beta} x_1^{\alpha_1} \cdots x_{d_1}^{\alpha_{d_1}} y_1^{\beta_1} \cdots y_{d_2}^{\beta_{d_2}},$$

where $\alpha_1 + \cdots + \alpha_{d_1} = \beta_1 + \cdots + \beta_{d_2} \geq 1$ for every (α,β) in $\mathbb{N}_{>0}^{d_1+d_2}$. By Corollary 4.8, $\int_{\mathbb{A}_k^{d_1}} i_1^* \mathcal{S}_{g_1}$ vanishes in $\mathcal{M}_{\mathbb{G}_{m,k}}^{\mathbb{G}_{m,k}}$, hence $\int_{\mathbb{A}_k^{d_1}} i_1^* A_{v_0}$ vanishes in $\mathcal{M}_{\mathbb{A}_k^1 \times_k \mathbb{G}_{m,k}}^{\mathbb{G}_{m,k}}$. Because the following diagram

$$
\begin{array}{ccc}
\mathcal{M}_{X_0(\mathbf{g}) \times_k \mathbb{A}_k^1 \times_k \mathbb{G}_{m,k}}^{\mathbb{G}_{m,k}} & \xrightarrow{\ \Psi_Q\ } & \mathcal{M}_{X_0(\mathbf{g}) \times_k \mathbb{G}_{m,k}}^{\mathbb{G}_{m,k}} \\
{\scriptstyle \int_{\mathbb{A}_k^{d_1}} i_1^*} \downarrow & & {\scriptstyle \int_{\mathbb{A}_k^{d_1}} i_1^*} \downarrow \\
\mathcal{M}_{\mathbb{A}_k^1 \times_k \mathbb{G}_{m,k}}^{\mathbb{G}_{m,k}} & \xrightarrow{\ \Psi_Q\ } & \mathcal{M}_{\mathbb{G}_{m,k}}^{\mathbb{G}_{m,k}}
\end{array}
$$

commutes, the lemma thus follows. $\square$

Let v be an abitrary rupture vertex of the tree of contact $\tau(f,0)$ and $a(v)$ the predecessor of v in the augmented set of rupture vertices. Then the polynomial Q_v is a factor of $Q_{a(v)}$. Suppose that $Q_v(X,1)$ has m disjoint zeroes in $\mathbb{A}_k^1$.

Lemma 5.3. *The equality $A_v = mA_{a(v)}$ holds in $\mathcal{M}_{X_0(\mathbf{g}) \times_k \mathbb{A}_k^1 \times_k \mathbb{G}_{m,k}}^{\mathbb{G}_{m,k}}$.*

Proof. We first notice that $Q_v^{-1}(0)$ is a smooth subvariety in $\mathbb{G}_{m,k} \times_k \mathbb{G}_{m,k}$, equivariant under a diagonal $\mathbb{G}_{m,k}$-action and that the second projection pr_2 of the product $\mathbb{A}_k^1 \times_k \mathbb{G}_{m,k}$ induces a homogeneous fibration $Q_v^{-1}(0) \to \mathbb{G}_{m,k}$. We denote by B_v the restriction of $A_{a(v)}$ above $Q_v^{-1}(0)$. Then, by [8], the element A_v in $\mathcal{M}_{X_0(\mathbf{g}) \times_k \mathbb{A}_k^1 \times_k \mathbb{G}_{m,k}}^{\mathbb{G}_{m,k}}$ is defined as the external product of the class of $id : \mathbb{A}_k^1 \to \mathbb{A}_k^1$ by the induced map $pr_2 : B_v \to \mathbb{G}_{m,k}$, which is diagonally monomial when restricted to $X_0(\mathbf{g}) \times_k \mathbb{G}_{m,k} \times_k \mathbb{G}_{m,k}$.

Consider the fibration $pr_2 : B_v \to \mathbb{G}_{m,k}$ defined by the composition of $B_v \to Q_v^{-1}(0)$ and $pr_2 : Q_v^{-1}(0) \to \mathbb{G}_{m,k}$. Then each fiber of $pr_2 : B_v \to \mathbb{G}_{m,k}$ is a disjoint union of m copies of a fiber of $A_{a(v)} \to \mathbb{A}_k^1 \times_k \mathbb{G}_{m,k}$ over one point (a,b) in $\mathbb{A}_k^1 \times_k \mathbb{G}_{m,k}$. It follows that $A_v = mA_{a(v)}$ in $\mathcal{M}_{X_0(\mathbf{g}) \times_k (\mathbb{A}_k^1 \times_k \mathbb{G}_{m,k})}^{\mathbb{G}_{m,k}}$. $\square$

It follows from Lemma 5.2 and Lemma 5.3 that $\int_{\mathbb{A}_k^{d_1}} i_1^* \Psi_{Q_v}(A_v) = 0$ for every (extended) rupture vertex v of $\tau(f,0)$. This completes the proof of Theorem 5.1. $\square$

Remark 5.4. In the case $f(x,y) = x + y$, the result can also be obtained directly from the Motivic Thom-Sebastiani Theorem (cf. [3, 4]).

5.2. In the following proposition, we prove the conjecture of Konsevich and Soibelman under some other conditions on $F = g$, namely assuming F is nondegenerate with respect to its Newton polyhedron Γ and no vertex of Γ lies in a coordinate plane.

Proposition 5.5. *Let g be a regular function on $\mathbb{A}_k^{d_1} \times_k \mathbb{A}_k^{d_2} \times_k \mathbb{A}_k^{d_3}$ such that $g(0,0,0) = 0$ and $g(tx, t^{-1}y, z) = g(x,y,z)$ for every t in $\mathbb{G}_{m,k}$ and (x,y,z) in $\mathbb{A}_k^{d_1} \times_k \mathbb{A}_k^{d_2} \times_k \mathbb{A}_k^{d_3}$. If g is nondegenerate with respect to its Newton polyhedron Γ and no vertex of Γ lies in a coordinate plane, then $\int_{\mathbb{A}_k^{d_1}} i_1^* \mathcal{S}_g$ vanishes in $\mathcal{M}_{\mathbb{G}_{m,k}}^{\mathbb{G}_{m,k}}$. In other words, Conjecture 1.1 is true in this case.*

Proof. Write the function g in the following form

$$g(x, y, z) = \sum_{(a,b,c)\in\mathbb{N}^d} g_{a,b,c} x^a y^b z^c,$$

where $d = d_1 + d_2 + d_3$, $g_{a,b,c}$ is in k, $x^a = x_1^{a_1} \cdots x_{d_1}^{a_{d_1}}$, ... The hypotheses on g require that, in the above expression of g, every (a, b, c) is in $\mathbb{N}_{>0}^d$ and $a_1 + \cdots + a_{d_1} = b_1 + \cdots + b_{d_2}$. The latter implies that, for each compact face γ of Γ, there exists a unique maximal subset M of $\{1, \ldots, d_1\}$ such that $\gamma + \mathbb{R}_{\geq 0}^M$ is a face of Γ, which is leant on γ, $\sigma_{\gamma,M}$ is contained in $\mathbb{R}_{\geq 0}^{d_1} \times \mathbb{R}_{>0}^{d_2+d_3}$, and M is nonempty (compare with Lemma 4.7). Similarly to the proof of Corollary 4.8, $\int_{\mathbb{A}_k^{d_1}} i_1^* \mathcal{S}_g$ vanishes in $\mathcal{M}_{\mathbb{G}_{m,k}}^{\mathbb{G}_{m,k}}$. Notice that, in this case, $h(z) = F(0, 0, z) = g(0, 0, z) = 0$, hence the local motivic Milnor fiber $\mathcal{S}_{h,0}$ also vanishes in $\mathcal{M}_{\mathbb{G}_{m,k}}^{\mathbb{G}_{m,k}}$. $\qquad\square$

5.3.   We consider now the more general case, $F(x, y, z) = g(x, y, z) + h(z)^N$, where g is as in Proposition 5.5, $h(z)$ is regular on $\mathbb{A}_k^{d_3}$ such that $h(0) = 0$, and N is a large enough natural number. By composition with the projection, we will view h as a function on $\mathbb{A}_k^d$.

We will use some following notations of [6]. Let $\eta : Y \to \mathbb{A}_k^d$ be a log-resolution of $(\mathbb{A}_k^d, Z)$ for Z a closed subset of $\mathbb{A}_k^d$ of codimension everywhere not less than 1. Assume that $\eta^{-1}(Z) = \bigcup_{i\in A} E_i$, with E_i the irreducible components of the divisor $\eta^{-1}(Z)$. If $\mathcal{J}_1$ is the sheaf of ideals defining a closed subset $Z_1 \subset Z$ and $\eta^{-1}(\mathcal{J}_1)\mathcal{O}_Y$ is locally principal, we define $N_i(\mathcal{J}_1)$, the multiplicity of $\mathcal{J}_1$ along E_i, by the equality of divisors

$$\eta^{-1}(Z_1) = \sum_{i\in A} N_i(\mathcal{J}_1) E_i.$$

Let Z_2 be another closed subset of codimension ≥ 1 such that, for its corresponding sheaf of ideals $\mathcal{J}_2$, $\eta^{-1}(\mathcal{J}_2)\mathcal{O}_Y$ is locally principal, and that $Z = Z_1 \cup Z_2$. Then we set

$$c_\eta(\mathcal{J}_1, \mathcal{J}_2) := \sup_{\{i\in A|N_i(\mathcal{J}_2)>0\}} \frac{N_i(\mathcal{J}_1)}{N_i(\mathcal{J}_2)}).$$

We define $c(\mathcal{J}_1, \mathcal{J}_2)$ as the infimum of all $c_\eta(\mathcal{J}_1, \mathcal{J}_2)$ for η, a log-resolution of $(\mathbb{A}_k^d, Z_1 \cup Z_2)$ such that $\eta^{-1}(\mathcal{J}_1)\mathcal{O}_Y$ and $\eta^{-1}(\mathcal{J}_2)\mathcal{O}_Y$ are locally principal.

Theorem 5.6. *Let $F(x, y, z) = g(x, y, z) + h(z)^N$, where g is as in Proposition 5.5, $h(z)$ is regular on $\mathbb{A}_k^{d_3}$ such that $h(0) = 0$, N is a natural number. If $N > c((g), (h))$, then the following formula*

$$\int_{\mathbb{A}_k^{d_1}} i_1^* \mathcal{S}_F = \mathbb{L}^{d_1} \mathcal{S}_{h^N,0}$$

holds in $\mathcal{M}_{\mathbb{G}_{m,k}}^{\mathbb{G}_{m,k}}$.

Proof. Let us denote by i and j the inclusion of $(X_0(g) \cap X_0(h)) \times_k \mathbb{G}_{m,k}$ in $X_0(g) \times_k \mathbb{G}_{m,k}$ and $X_0(F) \times_k \mathbb{G}_{m,k}$, respectively. By [6], Theorem 5.7, we have

$$j^* \mathcal{S}_F - i^* \mathcal{S}_g = \mathcal{S}_{h^N}([X_0(g)]) - \Psi_\Sigma(\mathcal{S}_{h^N}(\mathcal{S}_g)),$$

where Σ is the polynomial $x_1 + x_2$ (cf. [6], Section 5). Then we get

$$\int_{\mathbb{A}_k^{d_1}} i_1^* \mathcal{S}_F - \int_{\mathbb{A}_k^{d_1}} i_1^* \mathcal{S}_g = \int_{\mathbb{A}_k^{d_1}} i_1^* \mathcal{S}_{h^N}([X_0(g)]) - \int_{\mathbb{A}_k^{d_1}} i_1^* \Psi_\Sigma(\mathcal{S}_{h^N}(\mathcal{S}_g)).$$

Now, by Proposition 5.5, $\int_{\mathbb{A}_k^{d_1}} i_1^* \mathcal{S}_g = 0$. An analogue to the proof of Lemma 5.2 shows that $\int_{\mathbb{A}_k^{d_1}} i_1^* \Psi_\Sigma(\mathcal{S}_{h^N}(\mathcal{S}_g))$ vanishes. It deduces that

$$\int_{\mathbb{A}_k^{d_1}} i_1^* \mathcal{S}_F = \int_{\mathbb{A}_k^{d_1}} i_1^* \mathcal{S}_{h^N}([X_0(g)]) = \mathbb{L}^{d_1} \mathcal{S}_{h^N,0}$$

holds in $\mathcal{M}_{\mathbb{G}_{m,k}}^{\mathbb{G}_{m,k}}$, which completes the proof. $\qquad\square$

References

[1] J. Denef and F. Loeser, *Motivic Igusa zeta functions*, J. Algebraic Geom. **7** (1998), 505-537.
[2] J. Denef and F. Loeser, *Germs of arcs on singular algebraic varieties and motivic integration*, Invent. Math. **135** (1999), no. 1, 201-232.
[3] J. Denef and F. Loeser, *Motivic exponential integrals and a motivic Thom-Sebastiani theorem*, Duke Math. J. **99** (1999), no. 2, 285-309.
[4] J. Denef and F. Loeser, *"Geometry on arc spaces of algebraic varieties"* in European congress of Mathematics, Vol. 1 (Barcelona, 2000), Progr. Math. **201**, Birkhaüser, Basel, 2001, 327-348.
[5] G. Guibert, *Espaces d'arcs et invariants d'Alexander*, Comment. Math. Helv. **77** (2002), 783-820.
[6] G. Guibert, F. Loeser, M. Merle, *Iterated vanishing cycles, convolution, and a motivic analogue of the conjecture of Steenbrink*, Duke Math. J. **132** (2006), no. 3, 409-457.
[7] G. Guibert, F. Loeser, M. Merle, *Nearby cycles and composition with a nondegenerate polynomial*, International Mathematics Research Notices **31** (2005), 1873-1888.
[8] G. Guibert, F. Loeser, M. Merle, *Composition with a two variable function*, Math. Res. Lett. **16** No. 3 (2009), 439-448.
[9] K. Hoornaert, *Newton polyhedra and the poles of Igusa's local zeta function*, Bull. Belg. Math. Soc. **9** (2002), 589-606.
[10] M. Kontsevich, Y. Soibelman, *Stability structures, motivic Donaldson-Thomas invariants and cluster transformations*, arXiv: 0811.2435v1 [math.AG] 16 Nov 2008.
[11] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, New Jersey, 1970.
[12] B. Toën, *Derived Hall algebras*, arXiv: 0501343v4 [math.QA] 25 Apr 2006.

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